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Poisson brackets as the adelic limit of quantum commutators*

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Abstract. In this paper, we show that the adelic limit of the quantum commutator between operators gives a suitable generalization of the Poisson-bracket on p-adic phase space

1. Introduction

The first quantum model covering the field of p-adic number Q_p was considered by Beltrametti and Cassinelli [1], who investigated the problem of choosing a number field in quantum theories from the position of quantum logic. Subsequently, great interest in p-adic physics has appeared in research into string theory [2–7].

The main purpose of these p-adic string investigations was to describe the spacetime on Planck distances with the aid of the field of p-adic numbers Q_p in accordance with an old idea concerning violation of Archimedean axioms on Planck distances. The non-Archimedean number field Q_p can be a good mathematical tool for describing such a physical model. In these articles, the physical interpretation of such high-level models as p-adic strings became problematic and, hence, simpler models, such as p-adic quantum mechanics and field theory, were also investigated [8–12].

There are two main approaches to p-adic quantization. The first is based on complex-valued wavefunctions of the p-adic argument $\psi : Q_p^3 \to C$.

The second approach is based on wavefunctions of p-adic arguments, which assume values in some extensions of Q_p such as quadratic extensions or the field of complex p-adic numbers C_p; for the definition of this field see, for example, [13].

We are interested in the first approach by regarding the prime number p as a variable. We consider, in some sense, an adelic approach in which we first propose a quantization scheme by means of the calculus of pseudo-differential operators; this technique, which gives rise to the same results as the standard quantization technique when applied to real-number theory, is able to provide a procedure that we can also use in the p-adic approach.

Subsequently, we shall study the problem of defining derivatives on the functional space of Schwartz–Bruath (SB) maps from Q_p to C. This problem, which does not admit a direct solution, has already been confronted by Vladimirov in [14] where a definition of a *derivative map* was given. Here we propose a different definition of the *derivative* map by means of which we shall construct a generalized form of the classical Poisson bracket.

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Finally, by considering large p (that is by performing the adelic limit $p \to \infty$) we shall show that the most relevant part of the action of a quantum commutator (more precisely, the first term of the expansion in $1/p$) is given by the Poisson bracket we have just constructed.

The correspondence principle for standard quantum mechanics with a real argument and complex-valued wavefunctions can be written in the following form:

$$[f, g](Q, P) = \mathrm{i}\hbar\{f, g\}(Q, P) + \mathrm{o}(\hbar) \tag{1}$$

where

$$\{f, g\} = \frac{\partial f}{\partial P}\frac{\partial g}{\partial Q} - \frac{\partial f}{\partial Q}\frac{\partial g}{\partial P} \tag{2}$$

denotes the Poisson bracket on the phase space $\boldsymbol{R}^2$ where canonical coordinates (Q, P) are used and $[f, g](Q, P)$ is the symbol of the operator $\hat{f} \cdot \hat{g} - \hat{g} \cdot \hat{f}$; $\mathrm{o}(\hbar)$ deletes terms of order larger than one in $\hbar$.

This equation is considered to be the deformation law of commutator $[f, g]$ with respect to the small parameter $\hbar$. The first-order coefficient in $\hbar$ is the Poisson bracket. Equality (1) can be rewritten in the form

$$-\frac{\mathrm{i}}{\hbar}[f, g] = \{f, g\} + \mathrm{o}(\hbar^{\epsilon}) \qquad \epsilon > 0 \qquad \hbar \to 0. \tag{3}$$

Hence, the Poisson bracket $\{f, g\}$ is the limit of fraction $-\mathrm{i}[f, g]/\hbar$ when the deformation parameter $\hbar$ appoaches zero. We prefer to consider $\hbar$ in (1) as a deformation parameter not directly the Planck constant since this is a physical quantity and it is impossible to say that it approaches zero.

We wish to obtain an analogue of deformation law (1) for quantum mechanics for complex-valued wavefunctions defined on the p-adic space.

The main mathematical problem is the absence of a well defined derivative of the maps $f: \boldsymbol{Q}_p \to \boldsymbol{C}$. Indeed, it was proved in [15] that the unique maps $f: \boldsymbol{Q}_p \to \boldsymbol{C}$ that are differentiable in the Frechet sense are constant. This is why no Schrödinger-like representation was ever considered for p-adic quantum models; see [8] where Vladimirov and Volovich considered the Weyl representation to propose a quantization scheme. For this reason, we cannot construct a formula for the p-adic Poisson bracket in the usual way.

2. Schwartz–Bruhat maps and Fourier transforms

First, we give some details on p-adic numbers.

Let $\boldsymbol{Q}$ be the field of rational numbers; by means of the standard norm, we can complete it by obtaining the field of real numbers $\boldsymbol{R}$.

A different norm can be introduced on $\boldsymbol{Q}$: this is the p-adic norm; by completing the field $\boldsymbol{Q}$ with this norm we get the field of p-adic numbers $\boldsymbol{Q}_p$.

A famous theorem from number theory [13] tells us that the field $\boldsymbol{Q}$ can only be completed in these two ways.

Let p be a prime positive integer number ($p \neq 1$); for any non-zero rational number $x \in \boldsymbol{Q}$, there is a unique way of writing x as $x = p^{\nu} m/n$. Here m and n are integers which are not divisible by p while ν is an integer number. This equation is a trivial consequence of the decomposition of x into prime factors.

The p-adic norm is defined as

$$|x|_p = |p^\nu m/n|_p = p^{-\nu} \qquad |0|_p = 0 \tag{4}$$

and satisfies the *strong* triangular inequality

$$|x+y|_p \leqslant \max(|x|_p, |y|_p). \tag{5}$$

This norm is non-Archimedean; see [13].

If we complete $\boldsymbol{Q}$ with respect to $|\cdot|_p$, we obtain the field of p-adic numbers $\boldsymbol{Q}_p$.

For the convenience of the reader, we recall that any p-adic number can be uniquely written in the form

$$x = \sum_{k=-n}^{\infty} x_k p^k \tag{6}$$

where the numbers x_k are integers, $x_k = 0, 1, \ldots, p-1$. Here, the integer number n is not fixed but is a function of x. This expression is closely related to the usual decimal expression of a real number.

Now we introduce the adele group. (See [14] and the book by Gel'fand *et al* [16] for details.) An infinite sequence

$$x = (x_\infty, x_2, \ldots, x_p, \ldots) \tag{7}$$

is called an *adele* if $x_\infty \in \boldsymbol{R}$, $x_p \in \boldsymbol{Q}_p$ for all $p = 2, \ldots$ and all but a finite number of components are p-adic integers. The set A of all adeles is an additive group if the sums are performed component-wise. One can show that A is a locally compact topological space and the invariant (Haar) measure $\mathrm{d}x$ on A is given by

$$\mathrm{d}x = \mathrm{d}x_\infty \, \mathrm{d}x_2 \ldots \mathrm{d}x_p \ldots \tag{8}$$

where $\mathrm{d}x_p$ is the Haar measure on $\boldsymbol{Q}_p$ satisfying the properties

$$\int_{|x|_p \leqslant 1} \mathrm{d}x = 1 \qquad \mathrm{d}(ax) = |a|_p \, \mathrm{d}x. \tag{9}$$

Now we recall the definition of the space $\mathcal{D}$ of SB maps on A; they are the finite linear combinations of elementary maps of the form:

$$\phi(x) = \phi_\infty(x_\infty)\phi_2(x_2)\ldots\phi_p(x_p)\ldots \tag{10}$$

where

(i) $\phi_\infty(x_\infty)$ is an infinitely differentiable function on $\boldsymbol{R}$;

(ii) $\phi_p(x_p)$ are, for all p, finite and piecewise constant; and

(iii) for all p, except a finite number, $\phi_p(x_p) = 1$ when x_p is a p-adic integer and $\phi_p(x_p) = 0$ when x_p is not a p-adic integer.

We remark here that SB maps always vanish outside some balls $|x|_p \leqslant p^n$. In particular, they are continuous and integrable. The Fourier transform can be defined from $\mathcal{D} \to \mathcal{D}$. To this end, we define the character of a p-adic number x as

$$\chi_p(x) = \exp 2\pi \mathrm{i} \{x\} \tag{11}$$

where $\{x\}$ is the *fractional part* of x given by the negative degree part of the canonical expansion of x. The character has the following property: $\chi(x+y) = \chi(x)\chi(y)$. Now we define the character χ on A

$$\chi(x) = \exp(-2\pi \mathrm{i} x_\infty)\chi_2(x_2)\ldots\chi_p(x_p)\ldots \tag{12}$$

and define the Fourier transform of maps $\phi \in \mathcal{D}$

$$\mathcal{F}\phi(x) = \tilde{\phi}(x) = \int_A \phi(y)\chi(xy)\,\mathrm{d}y \tag{13}$$

which is again a $\mathcal{D}$ map. The inversion formula holds:

$$\phi(x) = \int_A \tilde{\phi}(y)\chi(-xy)\,\mathrm{d}y. \tag{14}$$

Here

$$\int_A \mathrm{d}y \equiv \int_R \mathrm{d}x_\infty \int_{Q_2} \mathrm{d}x_2 \ldots \int_{Q_p} \mathrm{d}x_p \ldots. \tag{15}$$

Further properties of the Fourier transforms for SB maps can be found in [14, 16].

3. The quantization scheme

Consider the space $Q_p \times Q_p$ and the couple (Q, P) as variables on this space. We denote by $\mathcal{D}(Q_p{}^2)$ the functional space of maps from $Q_p \times Q_p$ to C which are of SB type for both their coordinates.

Let $f(Q, P)$ be a *classical* function on phase space; we assume that $f \in \mathcal{D}(Q_p{}^2)$.

We now construct an operator $\hat{f}$ associated with f on the space of test functions $\mathcal{S}$ which is assumed to be the space of SB maps depending only on Q.

We put

$$(\hat{f}\phi)(Q) = \int_{Q_p\times Q_p} \mathrm{d}Q'\,\mathrm{d}P'\, f((1-\tau)Q + \tau Q', -P')\chi(P'(Q-Q'))\phi(Q'). \tag{16}$$

Parameter τ, in the classical framework, is related to the problem of operator order: important values are 0, 1, $\frac{1}{2}$; in this case we have the so-called $\hat{Q}\hat{P}$, $\hat{P}\hat{Q}$ and the symmetric or Weyl quantizations, respectively.

Let us return to the p-adic approach. The main result connected with this definition is as follows.

Theorem 3.1. Consider two cases.

(i) If $f = f(Q)$, then $\hat{f}$ is multiplicative: $\hat{f}\cdot\phi(Q) = f(Q)\phi(Q)$.

(ii) If $f = f(P)$, then $\hat{f}$ is multiplicative in momentum representation:

$$\hat{f}\cdot\phi(Q) = \int_{Q_p} \mathrm{d}P\,\tilde{\phi}(P)f(P)\chi(-QP). \tag{17}$$

The proof is straightforward; one has only to take into account the well known formula relating the Fourier transform of the identity map to the δ-Dirac distribution.

In this theorem, one can also recognize that the classical function f plays the role of the *symbol* of the quantum operator $\hat{f}$. Of course, this quantization procedure is the same as the standard procedure when $\hat{f}$ corresponds to a function f on phase space. Finally, we wish to note that the function f cannot be chosen to be a polynomial since it has to take its values in C.

Lemma 3.1. Let $\phi(x)$ be a test function of $\mathcal{D}$ and f a map in $\mathcal{D}(Q_p^2)$. Then the following formulae hold:

$$(\hat{f}\phi)(Q) = \int_{Q_p\times Q_p} \mathrm{d}\mu\,\mathrm{d}\nu\,\tilde{f}(\mu,\nu)\chi(-\mu Q - \mu\nu\tau)\phi(Q+\nu) \tag{18}$$

where

$$(\tilde{f})(\mu,\nu) = \int_{Q_p\times Q_p} \mathrm{d}Q\,\mathrm{d}P\, f(Q,P)\chi(\mu Q + \nu P). \tag{19}$$

The proof is a direct calculation.

By using this formula, we consider two functions f and g in $\mathcal{D}(Q_p^2)$ and compute the commutator $[\hat{g},\hat{f}]\phi(Q)$.

Lemma 3.2. Let $\phi(x)$ be a test function of $\mathcal{D}$ and f, g maps in $\mathcal{D}(Q_p^2)$. Then the following formulae hold:

$$\hat{U}\phi(Q) \equiv [\hat{g},\hat{f}]\phi(Q) = \int_{Q_p\times Q_p} \mathrm{d}\alpha\,\mathrm{d}\beta\,\chi(-\beta Q - \alpha\beta\tau)\phi(Q+\alpha)\mathcal{F}U(\beta,\alpha) \tag{20}$$

where the Fourier transform of the symbol of the commutator is

$$\begin{aligned}\mathcal{F}U(\beta,\alpha) = \int_{Q_p\times Q_p} & \mathrm{d}\mu\,\mathrm{d}\nu\,\tilde{f}(\mu,\nu)\tilde{g}(\beta-\mu,\alpha-\nu) \\ & \times \chi(-2\tau\mu\nu + \tau\mu\alpha + \tau\beta\nu + \mu\nu)[\chi(-\mu\alpha) - \chi(-\nu\beta)].\end{aligned} \tag{21}$$

Finally, by using the inverse Fourier-transform formula, one gets the following theorem providing the expression for the commutator symbol.

Theorem 3.2. The symbol of commutator $[\hat{g},\hat{f}]$ is

$$\begin{aligned}U(Q,P) = \int_{Q_p^4} & \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\alpha'\,\mathrm{d}\beta'\,\tilde{f}(\alpha,\beta)\tilde{g}(\alpha',\beta') \\ & \times \chi(\tau\alpha\beta' + \tau\beta\alpha')\chi(-P(\beta+\beta') - Q(\alpha+\alpha'))[\chi(-\beta'\alpha) - \chi(-\alpha'\beta)].\end{aligned} \tag{22}$$

The next section will be devoted to the study of a possible generalization of Poisson brackets.

4. On the definition of the Poisson brackets

Consider a map f in $\mathcal{D}(Q_p^2)$ and write its inverse Fourier formula as

$$f(Q,P) = \int_{Q_p\times Q_p} \mathrm{d}\alpha\,\mathrm{d}\beta\,\tilde{f}(\alpha,\beta)\chi(-Q\alpha)\chi(P\beta). \tag{23}$$

Let us try to introduce a derivative map for functions in $\mathcal{D}(Q_p^2)$.

Definition 4.1. Let h be a map in $\mathcal{D}(Q_p)$. For any bounded map $a: Q_p \to C$, the a-derivative of h with respect to its variable x is defined as

$$\left(\frac{\partial h}{\partial x}\right)_a \equiv \int_{Q_p} \mathrm{d}\alpha\, a(\alpha)\tilde{h}(\alpha)\chi(-x\alpha). \tag{24}$$

We remark that, by setting $a(x) = |x|_p$, we get the definition given by Vladimirov in [14] of the differential operator D.

If we consider an infinite-component functional vector $A = (a_1, a_2, \ldots)$, where the maps $a_i: Q_p \to C$ are bounded, we can generalize this formula to the *gradient case*

$$\left(\frac{\partial h}{\partial x}\right)_A \equiv \left(\frac{\partial h}{\partial x}\right)_{a_i} = \int_{Q_p} \mathrm{d}\alpha\, a_i(\alpha)\tilde{h}(\alpha)\chi(-x\alpha). \tag{25}$$

The relevant case is obtained with the choice $A(\alpha) = \{\alpha_1, \alpha_2, \ldots\}$, in which case we have

$$\left(\frac{\partial h}{\partial x}\right) \equiv \int_{Q_p} \mathrm{d}\alpha\, (\alpha_i)\tilde{h}(\alpha)\chi(-x\alpha). \tag{26}$$

Now we consider two maps g, f in $\mathcal{D}(Q_p^2)$ and compute

$$\left(\frac{\partial g}{\partial Q}\right) \circ \left(\frac{\partial f}{\partial P}\right) - \left(\frac{\partial f}{\partial Q}\right) \circ \left(\frac{\partial g}{\partial P}\right) = \int_{Q_p^4} \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\alpha'\, \mathrm{d}\beta'\, \tilde{f}(\alpha, \beta)\tilde{g}(\alpha', \beta')$$
$$\times \chi(-Q(\alpha + \alpha') - P(\beta + \beta'))((\alpha_i') \circ (\beta_i) - (\alpha_i) \circ (\beta_i')) \tag{27}$$

where $\circ$ denotes some map from $Q_p \times Q_p \to Q$ which has still to be determined.

5. The asymptotic expansion of the quantum commutator in $1/p$

In this section we try to state a correspondence principle and at the same time evaluate the exact value of the Poisson-bracket expression. In other words, we shall determine the forms of the maps A and $\circ$ of the previous section.

Our idea is as follows. From a physical point of view, p-adic field theories could be useful for describing quantum phenomena at the order of scale of the Planck length l_h (see [17] for a discussion). Indeed, the non-Archimedean structure of spacetime is connected with the non-localization of the gravitation measurement.

When the scale order of the phenomena is l_h, we think that p-adic numbers absolutely have to be used; on the other hand, for very large scale order, real numbers can be used. So we suspect that the deformation parameter in this theory is $1/p$ which has to be identified essentially with l_h.

Now we shall show that the commutator limit $l_h \to 0$ is the Poisson bracket by proving that the most relevant part of the expansion of the quantum commutator with respect to the deformation parameter $1/p$ is a possible definition of the Poisson bracket when the maps A and $\circ$ are suitably chosen.

Consider now the term

$$\chi(\tau\alpha\beta' + \tau\beta\alpha')[\chi(-\alpha\beta') - \chi(-\alpha'\beta)] \tag{28}$$

in equation (22), assume for simplicity that $\tau = 0$ and expand this term with respect to $1/p$. To this end we notice that, for every p-adic γ, by writing $\gamma = \sum_{-s}^{\infty} \gamma_i p^i$ (for some s) we have $\{\gamma\} = \sum_{-s}^{-1} \gamma_i p^i$ and, therefore,

$$\{\gamma\} = \frac{1}{p}\gamma_{-1} + \frac{1}{p^2}(\ldots). \tag{29}$$

Now we calculate, for large p,

$$\chi(-\alpha\beta') = \exp(2\pi \mathrm{i}\left\{-\alpha\beta'\right\}) = 1 + 2\pi \mathrm{i}/p(-\alpha\beta')_{-1} + \cdots. \tag{30}$$

Now we write the first term in the expansion in $1/p$ of the commutator symbol in the form ($\tau = 0$)

$$U^{(1)}(Q, P) = 2\pi \mathrm{i}/p \int_{Q_p^4} \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\alpha'\, \mathrm{d}\beta'\, \tilde{f}(\alpha, \beta)\tilde{g}(\alpha', \beta') \\ \times \chi(-P(\beta + \beta') - Q(\alpha + \alpha'))[(-\alpha\beta')_{-1} - (-\alpha'\beta)_{-1}]. \tag{31}$$

Now let us return to the Poisson-bracket definition. We fix the definition of the $\circ$ map as follows. For any couple of members $u, v \in \boldsymbol{Q}_p$, we set $u \circ v = -(uv)_{-1}$. In our specific case, we define

$$A(\alpha) \circ B(\beta) \equiv -(-\alpha\beta)_{-1}. \tag{32}$$

Now we consider Poisson brackets.

Definition 5.1. Let f, g be two maps of $\mathcal{D}(\boldsymbol{Q}_p^2)$. The Poisson bracket is defined as follows:

$$\{g, f\}(Q, P) = \left(\frac{\partial g}{\partial Q}\right) \circ \left(\frac{\partial f}{\partial P}\right) - \left(\frac{\partial f}{\partial Q}\right) \circ \left(\frac{\partial g}{\partial P}\right). \tag{33}$$

This definition can also be written in the form

$$\{g, f\}(Q, P) = \int_{Q_p^4} \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\alpha'\, \mathrm{d}\beta'\, \tilde{f}(\alpha, \beta)\tilde{g}(\alpha', \beta') \\ \times \chi(-Q(\alpha + \alpha') - P(\beta + \beta'))[(-\alpha\beta')_{-1} - (-\alpha'\beta)_{-1}]. \tag{34}$$

Now we look at the expression for the symbol of the operator $[\hat{g}, \hat{f}]$ in the $\hat{Q}\hat{P}$ quantization scheme. In so doing, we realize that the following expression holds:

$$U = \mathrm{Symb}[\hat{g}, \hat{f}] = 2\pi \mathrm{i}/p\{g, f\} + \mathrm{o}(1/p^2) \tag{35}$$

which is the promised analogue of the deformation of law (1).

We have therefore proved our main result:

Theorem 5.1. The first term of the expansion in $1/p$ of the symbol of the quantum commutator (for the $\hat{Q}\hat{P}$ quantization scheme) is the Poisson bracket of the previous definition

$$(\mathrm{Symb}[\hat{g}, \hat{f}])^{(1)} = 2\pi \mathrm{i}/p\{g, f\}. \tag{36}$$

This theorem furnishes us with the classical limit of p-adic quantum mechanics and can also be interpreted as a suggestion for linking the parameter p of $\boldsymbol{Q}_p$ to some scale factor like l_h.

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